Patterns of coexisting superconducting and particle-hole condensates

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Abstract. We have studied systematically the influence of particle-hole symmetric and asymmetric kinetic terms on the ordered phases that we may observe competing or coexisting in a tetragonal system. We show that there are precise patterns of triplets of ordered phases that are accessible (i.e. it is impossible to observe two of them without the third one). We found a systematic way to predict these patterns of states and tested it by identifying at least 16 different patterns of three order parameters that necessarily coexist in the presence of the kinetic terms. We show that there are two types of general equations governing the competition of all these triplets of order parameters and we provide them.

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1. Introduction

Almost all important functional materials undergo a pleiad of phases that under certain conditions may coexist. Controlling the parameters leading to coexistence is of primary importance as it can provide access to new intriguing phenomena and functionalities. Already in the early eighties, systematic theoretical investigations of the coexistence of two ordered electronic phases appeared [1, 2, 3, 4, 5, 6, 7, 8, 9], motivated essentially by the general problem of antiferromagnetic superconductivity that emerged in organic superconductors and heavy fermions. Numerous theoretical studies of the coexistence and competition of two phases continue to appear as the number of such experimental paradigms multiplies.

It has been a general conclusion from all the above studies that we must take into consideration *on the same footing the two order parameters* (OPs) that compete and may eventually coexist, otherwise we miss qualitatively new phenomena associated with this competition. However, as we shall show below, additional order parameters may coexist as well, and the need to include them is equally important. In fact, we will show that the particle-hole symmetric and asymmetric kinetic terms (KTs) of the hamiltonian, impose patterns of three order parameters (or triplets of order parameters) that are unavoidable. Whenever two of the order parameters coexist the third one appears as well. Therefore, we must necessarily consider all three order parameters simultaneously.

We have considered a tetragonal tight binding system and we have studied 16 cases of phase coexistence involving various types of ferromagnetism, density waves [10, 11, 12, 13, 14] and superconductivity [15]. In the case of a tetragonal lattice, the possible OPs that can be observed along with the kinetic terms are 63, generating an $SU(8)$ Lie algebra [16]. The OPs we have studied, were chosen among those of an $SO(8)$ subalgebra [17, 18] that describes only even parity order parameters and is relevant for the nearly half-filled case. We have observed that the kinetic terms impose phases that do not initially exist in the Hamiltonian. These phases ought to have been already included in the Hamiltonian from the very beginning in order to study the system consistently. We have performed this procedure in all cases that we have studied and we obtained in all cases the self-consistence equations of all the order parameters involving the induced order. We observed that the two initial order parameters, the induced order and the mixing kinetic term satisfy a system of self-consistence equations that entangles their dynamics. They constitute closed sets of order parameters that need to be treated on the same footing.

Through detailed examination of a number of systems with many phases emphasizing on the role of particle-hole symmetric and asymmetric kinetic terms, we managed to extract a simple empirical rule which helped us to predict patterns of OPs that coexist when the kinetic terms are properly taken into account. *According to this rule the matrix product of the matrix representations of the two initially coexisting order parameters and the mixing kinetic term yields the matrix representation of the induced phase. Equivalently, the matrix product of the three involved order parameters yields the mixing KT matrix.* In all cases that were selected according to this rule the predicted phase coexistence was confirmed.

We have to remark that although the above rule is quite expected if we want a non zero mean value for the induced phase, there is no way to be certain that if this rule stands for a specific set of order parameters then we must obtain the coexistence described above unless we do the calculations. Indeed, only by considering on a

suitable spinor formalism the relevant triplets of order parameters within a BCS like mean field approach and extracting self-consistent gap equations we were able to identify definitely that these states appear altogether as an unavoidable pattern.

2. Results

We consider all even parity OPs that are possible in a tetragonal system and may be relevant for discussing a number of heavy fermion materials as well as high-T_c cuprates. To describe in a unified way the coexistence of various OPs we need to introduce an eight component spinor formalism. We introduce the following spinor

$$\Psi^\dagger_{\boldsymbol{k}} = \begin{pmatrix} \alpha^\dagger_{\boldsymbol{k}\uparrow} & \alpha^\dagger_{\boldsymbol{k}\downarrow} & \alpha^\dagger_{\boldsymbol{k}+\boldsymbol{Q}\uparrow} & \alpha^\dagger_{\boldsymbol{k}+\boldsymbol{Q}\downarrow} & \alpha_{-\boldsymbol{k}\uparrow} & \alpha_{-\boldsymbol{k}\downarrow} & \alpha_{-\boldsymbol{k}-\boldsymbol{Q}\uparrow} & \alpha_{-\boldsymbol{k}-\boldsymbol{Q}\downarrow} \end{pmatrix} \quad (1)$$

where $\alpha_{\boldsymbol{k}s}/\alpha^\dagger_{\boldsymbol{k}s}$ are the destruction/creation operators of an electron of momentum $\boldsymbol{k}$ in the Reduced Brillouin zone and spin projection $s = \uparrow, \downarrow$. This enlargement of the spinor space allows the simultaneous description of ferromagnetism, zone center (zero Cooper-pair momentum) and staggered (finite Cooper-pair momentum) superconductivity, charge and spin density waves. The density waves and the staggered superconductivity are characterized by the wave-vector $\boldsymbol{Q} = (\pi, \pi)$ which is the best nesting vector close to half-filling. To work in this eight dimensional spinor space we consider a base formed by the Kronecker products of the unit matrix and three of the usual Pauli matrices τ_i , ρ_j, σ_k where $i, j, k = 1, 2, 3$.

The possible order parameters arising from the preceding spinor theory are $4 \times 4 \times 4 = 64$. If we demand that our Hamiltonian is traceless we are left with 63 order parameters (including K.T.) that constitute the generators of an $SU(8)$ spectrum generating algebra [16]. In the case of tetragonal systems close to half-filling, equivalence of the Brillouin zone points $(\pi, 0) \equiv ((-\pi, 0))$ and $(0, \pi) \equiv (0, -\pi)$ imposes that the order parameters have even parity. The OPs satisfying this constraint are 28 (including K.T.) and form an $SO(8)$ spectrum generating algebra [17, 18]. The OPs that we have considered in this study were chosen among those 28 identified in Table 1 with their symbols adopted here.

Table 1. The 28 OPs that form an $SO(8)$ spectrum generating algebra and would be accessible in a tetragonal system close to half filling. In the next sections we demonstrate that particle-hole asymmetric and symmetric kinetic terms impose various patterns of triplets of the following OPs.

Order Parameter	***Type***
γ	nearest neighbours hopping term
δ	next nearest neighbours hopping term
$F_{x,y,z}$	ferromagnet along x,y,z-axis
$A_{x,y,z}$	d-wave ferromagnet along x,y,z-axis
W	charge density wave
J_c	orbital anti-ferromagnet
$M_{x,y,z}$	spin density wave along x,y,z-axis
$J^s_{x,y,z}$	spin nematic along x,y,z-axis
Δ_s	s-wave SC ($\boldsymbol{q} = 0$)
Δ_d	d-wave SC ($\boldsymbol{q} = 0$)
η	s-wave SC ($\boldsymbol{q} = \boldsymbol{Q}$)
$\Pi_{x,y,z}$	d-wave SC along x,y,z-axis ($\boldsymbol{q} = \boldsymbol{Q}$)

We note that in Table 1 there are 16 OPs corresponding to particle-hole condensates including the KTs and 12 superconducting states (including their complex conjugates). Moreover, 8 of the 12 superconducting OPs represent staggered SC in which the pairs have a finite total center-of-mass momentum ($\boldsymbol{q} = \boldsymbol{Q}$) bearing similarities to the Fulde-Ferrel states [19]. These quite exotic states are superconducting states with modulated superfluid density and as we will show below, they should play a crucial role in any antiferromagnetic SC state.

We report here 16 different patterns of OPs that are imposed by the particle-hole symmetric and asymmetric kinetic terms. They can be classified into 3 different types of OP mixing that according to their properties can be merged into two general groups. In all these cases we present the typical system of self-consistence equations that provide the OPs and we identify the kinetic terms that are responsible for the OPs mixing.

2.1. First type of OPs mixing

In this first case, we consider that the Hamiltonian consists of the two kinetic terms and three order parameters. These order parameters have been chosen according to the empirical rule mentioned in the introduction i.e. the matrix product of the three order parameters yields the kinetic term that causes their mixing. In Table 2 we present the different combinations that fall into this class.

Table 2. Triplets of order parameters that form patterns imposed by the kinetic terms mentioned in the last column. In all cases we have the same system of self-consistence equations for the OPs, provided we replace the corresponding OPs of the same column.

***OP** 1*	***OP** 2*	***OP** 3*	***mixing KT***
M_z	Δ_d	Π_z	δ
W	Δ_s	η	δ
J_y^s	Δ_s	Π_y	δ
Π_z	Δ_s	M_z	γ
η	Δ_d	W	γ
Π_y	Δ_d	J_y^s	γ

We consider explicitly the first combination of the preceding table in order to demonstrate the general equations governing the phase coexistence and competition in the above patterns. In fact, the self-consistence equations that result are the same for all six patterns provided we replace the corresponding OPs that are in the same column. The Hamiltonian corresponding to the first case is given by the relation

$$H = \sum_{\boldsymbol{k}} \Psi_{\boldsymbol{k}}^{\dagger} \left(\gamma\tau_3\rho_3 + \delta\tau_3 - M_z\tau_3\rho_1\sigma_3 - \Pi_z\tau_2\rho_2\sigma_1 + \Delta_d\tau_2\rho_3\sigma_2 \right) \Psi_{\boldsymbol{k}} \tag{2}$$

where we have suppressed the momentum index $\boldsymbol{k}$ and the Kronecker product's symbol $\otimes$. The energy eigenvalues are

$$E_{\pm} = \sqrt{M_z^2 + \gamma^2 + \delta^2 + \Pi_z^2 + \Delta_d^2 \pm 2\sqrt{(M_z^2+\gamma^2)\delta^2 - 2\delta M_z\Pi_z\Delta_d + (\Delta_d^2+\gamma^2)\Pi_z^2}} \tag{3}$$

The self-consistence equations of the order parameters are the following

$$
\begin{aligned}
M_z = \frac{1}{4}\sum_{\boldsymbol{k}'}\frac{V^{M_z}_{\boldsymbol{k}\boldsymbol{k}'}}{E_+^2-E_-^2}\Bigg\{ & \frac{M_z\left(E_+^2-E_-^2+4\delta^2\right)-4\delta\Delta_d\Pi_z}{E_+}\tanh\left(\frac{E_+}{2T}\right) \\
& +\frac{M_z\left(E_+^2-E_-^2-4\delta^2\right)+4\delta\Delta_d\Pi_z}{E_-}\tanh\left(\frac{E_-}{2T}\right)\Bigg\}
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
\Delta_d = \frac{1}{4}\sum_{\boldsymbol{k}'}\frac{V^{\Delta_d}_{\boldsymbol{k}\boldsymbol{k}'}}{E_+^2-E_-^2}\Bigg\{ & \frac{\Delta_d\left(E_+^2-E_-^2+4\Pi_z^2\right)-4\delta M_z\Pi_z}{E_+}\tanh\left(\frac{E_+}{2T}\right) \\
& +\frac{\Delta_d\left(E_+^2-E_-^2-4\Pi_z^2\right)+4\delta M_z\Pi_z}{E_-}\tanh\left(\frac{E_-}{2T}\right)\Bigg\}
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
\Pi_z = \frac{1}{4}\sum_{\boldsymbol{k}'}\frac{V^{\Pi_z}_{\boldsymbol{k}\boldsymbol{k}'}}{E_+^2-E_-^2}\Bigg\{ & \frac{\Pi_z\left(E_+^2-E_-^2+4\gamma^2+4\Delta_d^2\right)-4\delta\Delta_d M_z}{E_+}\tanh\left(\frac{E_+}{2T}\right) \\
& +\frac{\Pi_z\left(E_+^2-E_-^2-4\gamma^2-4\Delta_d^2\right)+4\delta\Delta_d M_z}{E_-}\tanh\left(\frac{E_-}{2T}\right)\Bigg\}
\end{aligned} \tag{6}
$$

where the OPs and KTs depend on $\boldsymbol{k}'$. The mixing role of the kinetic term is explicit already in the form of the equation. In fact, the usual BCS equations for each one of the order parameters are expected to have the general form

$$\Delta_{\boldsymbol{k}} = \sum_{\boldsymbol{k}'} f(E_{\boldsymbol{k}'},T)V_{\boldsymbol{k}\boldsymbol{k}'}\Delta_{\boldsymbol{k}'} \tag{7}$$

where f is a function of the energy dispersion E and temperature T. Each BCS equation supports solutions of zero and non-zero order parameter, depending on the temperature. On the other hand, in our case each OP self-consistence equation has the general form

$$\Delta_{\boldsymbol{k}} = \sum_{\boldsymbol{k}'} V_{\boldsymbol{k},\boldsymbol{k}'}\left\{f(E_{\boldsymbol{k}'},T)\Delta_{\boldsymbol{k}'}+g(E_{\boldsymbol{k}'},T)m_{\boldsymbol{k}'}A_{\boldsymbol{k}'}B_{\boldsymbol{k}'}\right\} \tag{8}$$

where f, g are function of the energy dispersion and temperature, $m_{\boldsymbol{k}}$ is the mixing kinetic term and $A_{\boldsymbol{k}}$ and $B_{\boldsymbol{k}}$ are the other two OPs. *We observe that a solution of zero order parameter is not possible unless the mixing term or one at least of the other order parameters is also zero.* ***This suggests that in the presence of the mixing kinetic term we cannot have two order parameters without the third.*** *The three order parameters and the mixing term constitute a group that must be treated as an independent subsystem on the same footing.*

It is interesting to obtain the self-consistence equations when the kinetic term that does not contribute to the mixing is set to zero. In this case, the eigenenergies obtain the form

$$E_+ = \sqrt{(M_z+\delta)^2+(\Delta_d-\Pi_z)^2} \tag{9}$$

$$E_- = \sqrt{(M_z-\delta)^2+(\Delta_d+\Pi_z)^2} \tag{10}$$

while the first self-consistence equation becomes

$$M_z = \sum_{\boldsymbol{k}'} V^{M_z}_{\boldsymbol{k}\boldsymbol{k}'} \left\{ \frac{M_z + \delta}{\sqrt{(M_z + \delta)^2 + (\Delta_d - \Pi_z)^2}} \tanh\left(\frac{E_+}{2T}\right) \right.$$
$$\left. + \frac{M_z - \delta}{\sqrt{(M_z - \delta)^2 + (\Delta_d + \Pi_z)^2}} \tanh\left(\frac{E_-}{2T}\right) \right\} \tag{11}$$

2.2. Second type of OPs mixing

The second case involves a different coexistence pattern involving once again three order parameters and a mixing kinetic term. The following table contains the combinations belonging in this class:

Table 3. Same as in Table 2

OP 1	***OP 2***	***OP 3***	***mixing KT***
F_z	Π_x	Π_y	δ
A_z	η	Π_z	δ
F_y	J^s_x	M_z	γ
A_z	W	M_z	γ

Once again we present the typical results of one of these cases, specifically we consider the first. The Hamiltonian is

$$H = \sum_{\boldsymbol{k}} \Psi^\dagger_{\boldsymbol{k}} \left(\gamma\tau_3\rho_3 + \delta\tau_3 - \Pi_y\tau_2\rho_2 - \Pi_x\tau_2\rho_2\sigma_3 - F_z\tau_3\sigma_3\right) \Psi_{\boldsymbol{k}} \tag{12}$$

The corresponding quasi-particle poles are

$$E_{\pm+} = \gamma \pm \sqrt{\left(F_z - \delta\right)^2 + \left(\Pi_x + \Pi_y\right)^2} \tag{13}$$
$$E_{\pm-} = \gamma \pm \sqrt{\left(F_z + \delta\right)^2 + \left(\Pi_x - \Pi_y\right)^2} \tag{14}$$

Finally, we obtain the self-consistence equations

$$F_z = \frac{1}{8} \sum_{\boldsymbol{k}'} V^{F_z}_{\boldsymbol{k}\boldsymbol{k}'} \left\{ \frac{F_z - \delta}{\sqrt{\left(F_z - \delta\right)^2 + \left(\Pi_x + \Pi_y\right)^2}} \left[\tanh\left(\frac{E_{++}}{2T}\right) - \tanh\left(\frac{E_{-+}}{2T}\right)\right] \right.$$
$$\left. + \frac{F_z + \delta}{\sqrt{\left(F_z + \delta\right)^2 + \left(\Pi_x - \Pi_y\right)^2}} \left[\tanh\left(\frac{E_{+-}}{2T}\right) - \tanh\left(\frac{E_{--}}{2T}\right)\right] \right\} \tag{15}$$

$$\Pi_x = \frac{1}{8}\sum_{\boldsymbol{k'}} V^{\Pi_x}_{\boldsymbol{kk'}} \left\{ \frac{\Pi_x + \Pi_y}{\sqrt{\Big(\Pi_x + \Pi_y\Big)^2 + \Big(F_z - \delta\Big)^2}} \left[\tanh\left(\frac{E_{++}}{2T}\right) - \tanh\left(\frac{E_{-+}}{2T}\right)\right] \right.$$
$$\left. + \frac{\Pi_x - \Pi_y}{\sqrt{\Big(\Pi_x - \Pi_y\Big)^2 + \Big(F_z + \delta\Big)^2}} \left[\tanh\left(\frac{E_{--}}{2T}\right) - \tanh\left(\frac{E_{+-}}{2T}\right)\right] \right\} \tag{16}$$

$$\Pi_y = \frac{1}{8}\sum_{\boldsymbol{k'}} V^{\Pi_y}_{\boldsymbol{kk'}} \left\{ \frac{\Pi_x + \Pi_y}{\sqrt{\Big(\Pi_x + \Pi_y\Big)^2 + \Big(\delta - F_z\Big)^2}} \left[\tanh\left(\frac{E_{++}}{2T}\right) - \tanh\left(\frac{E_{-+}}{2T}\right)\right] \right.$$
$$\left. + \frac{\Pi_x - \Pi_y}{\sqrt{\Big(\Pi_x - \Pi_y\Big)^2 + \Big(F_z + \delta\Big)^2}} \left[\tanh\left(\frac{E_{+-}}{2T}\right) - \tanh\left(\frac{E_{--}}{2T}\right)\right] \right\} \tag{17}$$

As one can observe, the above equations do not have the general form (8) in which the mixing role of the relevant kinetic term is explicit as in the previous case. However, by performing a Taylor expansion with respect to the order parameters up to quadratic order terms we can show that such a relation does exist. These self-consistence equations imply once again that if the mixing term is present we cannot have the two order parameters without the third.

Indeed, let us consider (15) supposing that F_z is absent from the initial Hamiltonian. Then we set $F_z = 0$ in the right side of (15) and this equation will now provide the *induced* part of F_z:

$$F_z^{\,induced} \sim \sum_{\boldsymbol{k'}} \left\{ \frac{-\delta}{\sqrt{\delta^2 + \Big(\Pi_x + \Pi_y\Big)^2}} \left[\tanh\left(\frac{E'_{++}}{2T}\right) - \tanh\left(\frac{E'_{-+}}{2T}\right)\right] \right.$$
$$\left. + \frac{\delta}{\sqrt{\delta^2 + \Big(\Pi_x - \Pi_y\Big)^2}} \left[\tanh\left(\frac{E'_{+-}}{2T}\right) - \tanh\left(\frac{E'_{--}}{2T}\right)\right] \right\} \tag{18}$$

where we have introduced the new energy dispersions E', by setting $F_z = 0$:

$$E'_{\pm+} = \gamma \pm \sqrt{\delta^2 + \Big(\Pi_x + \Pi_y\Big)^2} \tag{19}$$

$$E'_{\pm-} = \gamma \pm \sqrt{\delta^2 + \Big(\Pi_x - \Pi_y\Big)^2} \tag{20}$$

A zero induced F_z term is expected if one of the two following conditions holds. On one hand, we may have $\delta = 0$, in which case we confirm that we will not have induced F_z if the mixing kinetic term vanishes. On the other hand, we may have $E_{\pm+} = E_{\mp+}$ and $E_{\pm-} = E_{\mp-}$. This last condition can be realized only when $\Pi_x = 0$

or $\Pi_y = 0$. *Consequently we conclude that if the mixing term is present and two order parameters are non zero we have an induced order* F_z. *i.e. the three phases coexist.*

It is instructive to derive here as well, the self-consistence equations when the irrelevant, to the mixing, kinetic term vanishes.

$$\tilde{E}_{\pm +} = E^{\gamma=0}_{\pm +} = \pm\sqrt{\Big(F_z - \delta\Big)^2 + \Big(\Pi_x + \Pi_y\Big)^2} \tag{21}$$

$$\tilde{E}_{\pm -} = E^{\gamma=0}_{\pm -} = \pm\sqrt{\Big(F_z + \delta\Big)^2 + \Big(\Pi_x - \Pi_y\Big)^2} \tag{22}$$

We observe that $\tilde{E}_{+\pm} = -\tilde{E}_{-\mp}$. This equality simplifies the self-consistence equations. For example we have

$$F_z = \frac{1}{4}\sum_{\mathbf{i}'} V^{F_z}_{\boldsymbol{k}\boldsymbol{k}'} \left\{ \frac{F_z - \delta}{\sqrt{\Big(F_z - \delta\Big)^2 + \Big(\Pi_x + \Pi_y\Big)^2}} \tanh\left(\frac{\tilde{E}_{++}}{2T}\right) \right.$$
$$\left. + \frac{F_z + \delta}{\sqrt{\Big(F_z + \delta\Big)^2 + \Big(\Pi_x - \Pi_y\Big)^2}} \tanh\left(\frac{\tilde{E}_{+-}}{2T}\right) \right\} \tag{23}$$

Quite remarkably, we have encountered the same form of self-consistence equation in Section 2.1 when the non mixing kinetic term was set to zero. *This common feature reveals that these two cases share the same mixing "mechanism", constituting specific examples of a more general coexistence pattern.*

The next pattern that we shall discuss here is the coexistence of two specific phases, s-wave and d-wave SC OPs in the presence of the two kinetic terms, *where the kinetic terms play both the role of the mixing terms and the OPs at the same time.* As far as the form of the equations that we derive, they belong to the same general coexistence pattern like the one reported just above. The Hamiltonian is

$$H = \sum_{\boldsymbol{k}} \Psi^\dagger_{\boldsymbol{k}} \Big(\gamma\tau_3\rho_3 + \delta\tau_3 - \Delta_d\tau_2\rho_3\sigma_2 - \Delta_s\tau_2\sigma_2\Big)\Psi_{\boldsymbol{k}} \tag{24}$$

The poles of the Green's function are

$$E_\pm = \sqrt{\Big(\Delta_s \pm \Delta_d\Big)^2 + (\gamma \pm \delta)^2} \tag{25}$$

It is evident that they have the form of the previous cases Section 2.1 and Section 2.2 when we set the irrelevant kinetic terms equal to zero. The self-consistence equations obey the same rule

$$\Delta_d = \frac{1}{4}\sum_{\boldsymbol{k}'} V^{\Delta_d}_{\boldsymbol{k}\boldsymbol{k}'} \left\{ \frac{\Delta_d + \Delta_s}{\sqrt{\Big(\Delta_s + \Delta_d\Big)^2 + (\gamma+\delta)^2}} \tanh\left(\frac{E_+}{2T}\right) \right.$$

$$+\frac{\Delta_d-\Delta_s}{\sqrt{\left(\Delta_s-\Delta_d\right)^2+(\gamma-\delta)^2}}\tanh\left(\frac{E_-}{2T}\right)\Bigg\}$$

$$\Delta_s=\frac{1}{4}\sum_{\boldsymbol{k}'}V^{\Delta_s}_{\boldsymbol{k}\boldsymbol{k}'}\Bigg\{\frac{\Delta_s+\Delta_d}{\sqrt{\left(\Delta_s+\Delta_d\right)^2+(\gamma+\delta)^2}}\tanh\left(\frac{E_+}{2T}\right)$$

$$+\frac{\Delta_s-\Delta_d}{\sqrt{\left(\Delta_s-\Delta_d\right)^2+(\gamma-\delta)^2}}\tanh\left(\frac{E_-}{2T}\right)\Bigg\} \qquad (26)$$

2.3. Third type of OPs mixing

The next case we consider has distinct properties from the preceding encountered in Section 2.1 and Section 2.2. We have found that the following combinations all have the same coexistence pattern.

Table 4. Distinct type of mixing compared to the one related to Section 2.1 and Section 2.2. The following combinations obey the same system of self-consistence equations.

OP 1	***OP 2***	***OP 3***	***mixing KT***
F_z	W	M_z	δ
F_z	J_c	J^s_z	δ
A_x	J^s_y	M_z	δ
F_z	η	Π_z	γ

The example in this type of mixing is the first of Table 4, which is described by the Hamiltonian

$$H=\sum_{\boldsymbol{k}}\Psi^\dagger_{\boldsymbol{k}}\left(\gamma\tau_3\rho_3+\delta\tau_3-M_z\tau_3\rho_1\sigma_3-W\tau_3\rho_1-F_z\tau_3\sigma_3\right)\Psi_{\boldsymbol{k}} \qquad (27)$$

with corresponding eigenenergies

$$E_{\pm+}=(F_z\mp\delta)+\sqrt{\left(M_z\pm W\right)^2+\gamma^2} \qquad (28)$$

$$E_{\pm-}=(F_z\mp\delta)-\sqrt{\left(M_z\pm W\right)^2+\gamma^2} \qquad (29)$$

We observe that the structure of the poles are different from the ones found in Sections 2.1 and 2.2. The self-consistence equations are given from the following relations

$$F_z=\frac{1}{8}\sum_{\boldsymbol{k}'}V^{F_z}_{\boldsymbol{k}\boldsymbol{k}'}\left\{\tanh\left(\frac{E_{++}}{2T}\right)+\tanh\left(\frac{E_{+-}}{2T}\right)+\tanh\left(\frac{E_{-+}}{2T}\right)+\tanh\left(\frac{E_{--}}{2T}\right)\right\} \qquad (30)$$

$$M_z = \frac{1}{8}\sum_{\boldsymbol{k}'} V^{M_z}_{\boldsymbol{k}\boldsymbol{k}'} \left\{ \frac{M_z + W}{\sqrt{\Big(M_z + W\Big)^2 + \gamma^2}} \left[\tanh\left(\frac{E_{++}}{2T}\right) - \tanh\left(\frac{E_{+-}}{2T}\right)\right] \right.$$

$$\left. + \frac{M_z - W}{\sqrt{\Big(M_z - W\Big)^2 + \gamma^2}} \left[\tanh\left(\frac{E_{-+}}{2T}\right) - \tanh\left(\frac{E_{--}}{2T}\right)\right] \right\} \tag{31}$$

$$W = \frac{1}{8}\sum_{\boldsymbol{k}'} V^{W}_{\boldsymbol{k}\boldsymbol{k}'} \left\{ \frac{M_z + W}{\sqrt{\Big(M_z + W\Big)^2 + \gamma^2}} \left[\tanh\left(\frac{E_{++}}{2T}\right) - \tanh\left(\frac{E_{+-}}{2T}\right)\right] \right.$$

$$\left. + \frac{M_z - W}{\sqrt{\Big(M_z - W\Big)^2 + \gamma^2}} \left[\tanh\left(\frac{E_{--}}{2T}\right) - \tanh\left(\frac{E_{-+}}{2T}\right)\right] \right\} \tag{32}$$

We have to remark that Equations (31) and (32) have similarities with the results of the previous sections. Though, Equation (30) is totally different. Close observation of the equation and the eigenenergies, shows that great simplification occurs when $\gamma = 0$. In this case we have

$$E_{\pm+} = (F_z \mp \delta) + (M_z \pm W) \tag{33}$$

$$E_{\pm-} = (F_z \mp \delta) - (M_z \pm W) \tag{34}$$

and as far as the self-consistence equations are concerned

$$F_z = \frac{1}{8}\sum_{\boldsymbol{k}'} V^{F_z}_{\boldsymbol{k}\boldsymbol{k}'} \left\{ \tanh\left(\frac{E_{++}}{2T}\right) + \tanh\left(\frac{E_{+-}}{2T}\right) + \tanh\left(\frac{E_{-+}}{2T}\right) + \tanh\left(\frac{E_{--}}{2T}\right) \right\} \tag{35}$$

$$M_z = \frac{1}{8}\sum_{\boldsymbol{k}'} V^{M_z}_{\boldsymbol{k}\boldsymbol{k}'} \left\{ \tanh\left(\frac{E_{++}}{2T}\right) - \tanh\left(\frac{E_{+-}}{2T}\right) + \tanh\left(\frac{E_{-+}}{2T}\right) - \tanh\left(\frac{E_{--}}{2T}\right) \right\} \tag{36}$$

$$W = \frac{1}{8}\sum_{\boldsymbol{k}'} V^{W}_{\boldsymbol{k}\boldsymbol{k}'} \left\{ \tanh\left(\frac{E_{++}}{2T}\right) - \tanh\left(\frac{E_{+-}}{2T}\right) - \tanh\left(\frac{E_{-+}}{2T}\right) + \tanh\left(\frac{E_{--}}{2T}\right) \right\} \tag{37}$$

According to what he have been taught from the previous sections, the symmetry the above equations present, implies that we have reached to a triplet of order parameters that necessarily coexist in the presence of the corresponding mixing kinetic term. This can be shown as follows. Any of these order parameters can be zero only if the spectrum is particle-hole symmetric. This occurs *only* when two out of the four terms are zero. *Consequently if three of these terms are non zero the fourth will be non zero, too.*

The final case we present, has two order parameters and the two kinetic terms. As we shall see it belongs to the same coexistence pattern of the above cases. The Hamiltonian is

$$H = \sum_{\boldsymbol{k}} \Psi^\dagger_{\boldsymbol{k}} \left(\gamma\tau_3\rho_3 + \delta\tau_3 + A_x\tau_3\rho_3\sigma_1 + F_x\tau_3\sigma_1 \right) \Psi_{\boldsymbol{k}} \tag{38}$$

The eigenenergies are equal to

$$E_{\pm+} = (A_x \pm F_x) + (\delta \pm \gamma) \tag{39}$$

$$E_{\pm-} = (A_x \pm F_x) - (\delta \pm \gamma) \tag{40}$$

The corresponding self-consistence equations are

$$A_x = \frac{1}{8}\sum_{\boldsymbol{k}'} V^{A_x}_{\boldsymbol{k}\boldsymbol{k}'} \left\{ \tanh\left(\frac{E_{++}}{2T}\right) + \tanh\left(\frac{E_{+-}}{2T}\right) + \tanh\left(\frac{E_{-+}}{2T}\right) + \tanh\left(\frac{E_{--}}{2T}\right) \right\} \tag{41}$$

$$F_x = \frac{1}{8}\sum_{\boldsymbol{k}'} V^{F_x}_{\boldsymbol{k}\boldsymbol{k}'} \left\{ \tanh\left(\frac{E_{++}}{2T}\right) - \tanh\left(\frac{E_{+-}}{2T}\right) + \tanh\left(\frac{E_{-+}}{2T}\right) - \tanh\left(\frac{E_{--}}{2T}\right) \right\} \tag{42}$$

3. Discussion

Having studied 16 cases of coexisting OPs in the presence of the particle-hole symmetric and asymmetric KTs, we have found 3 different types of coexistence patterns. The 11 cases studied in Sections 2.1 and 2.2 seem to have the same type of coexistence "mechanism" belonging to a more general *Coexistence Scheme*. On the other hand the 5 cases that we presented in Section 2.3 originate by a distinct coexistence "mechanism". **Consequently, we conclude that the 16 cases we studied merge into two general *Coexistence Schemes*** as in Table 5.

Moreover, we have found that special coexistence patterns can be observed even when he have only 2 OPs. In this case the kinetic terms play a dual role. They behave as the mixing KTs and the members of triplets of coexisting OPs. In that case, with both kinetic terms present we cannot observe one of the two OPs without the second one. For example, in the presence of both kinetic terms, d-wave SC coexists with s-wave SC ! This shows that the generation of these patterns is not a special property owned by the KTs. **Consequently, these *Coexistence Schemes* originate due to more general relations that are satisfied by quartets of the $\boldsymbol{SU(8)}$ generators.** We expect that other terms, apart from the KTs could play the role of the mixing terms and produce different type of coexistence patterns. We will present elsewhere a complete account of all patterns of coexisting states that correspond to the above mentioned quartets.

Finally, apart from the general conclusions that we inferred about phase coexistence, we obtained valuable results concerning specific coexisting triplets of OPs that may correspond to the physical situation in numerous correlated systems of interest. Particularly, we have observed that:

Table 5. *The two distinct* ***Coexistence Schemes***

Scheme	***OP** 1*	***OP** 2*	***OP** 3*	***mixing KT***
I	M_z	Δ_d	Π_z	δ
I	W	Δ_s	η	δ
I	J^s_y	Δ_s	Π_y	δ
I	Π_z	Δ_s	M_z	γ
I	η	Δ_d	W	γ
I	Π_y	Δ_d	J^s_y	γ
I	F_z	Π_x	Π_y	δ
I	A_z	η	Π_z	δ
I	F_y	J^s_x	M_z	γ
I	A_z	W	M_z	γ
I	Δ_s	Δ_d	δ	γ
II	F_z	W	M_z	δ
II	F_z	J_c	J^s_z	δ
II	A_x	J^s_y	M_z	δ
II	F_z	η	Π_z	γ
II	F_x	A_x	δ	γ

- Density waves, zone-center superconductivity ($\boldsymbol{q} = \mathbf{o}$) and staggered superconductivity ($\boldsymbol{q} = \boldsymbol{Q}$) constitute a triplet of OPs that necessarily coexist in the presence of the KTs. Such an observation is of general relevance for all antiferromagnetic superconductors, a category of materials that includes organics, heavy fermions, high-T_c cuprates etc.
- Ferromagnetism, charge density waves and spin density waves constitute another triplet of OPs that necessarily coexist in the presence of the asymmetric KT. This observation has already been reported before and shown to be related with the colossal magnetoresistance phenomenon [20].
- s-wave and d-wave superconducting OPs always coexist in the combined presence of the symmetric and asymmetric KTs. Needless to note that high-T_c cuprates as well as numerous heavy fermion systems are believed to be d-wave SC. Our observations imply that a pure d-wave SC state is an oversimplification.
- s-wave and d-wave ferromagnetic OPs always coexist in the combined presence of the symmetric and asymmetric KTs. The implications of this observation need to be investigated.

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